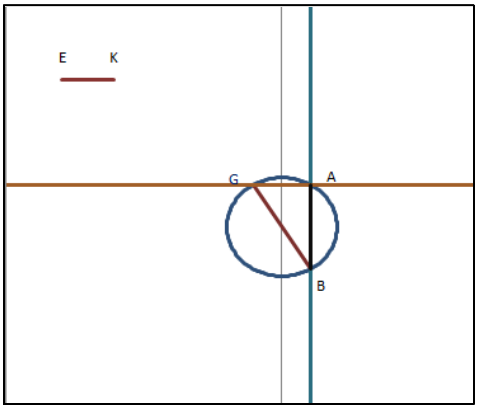
Excel files for Al Haytham's problem

By Pavlos Michas, email: pmichas@eled.duth.gr

Al Haytham proposed the problem: Given a spherical reflecting surface and two fixed points not on this surface to find the point o the surface at which light from one of the fixed points will be reflected on the other. He investigated the number of points who permit this to happen (Mihas 2002). He found at most 4 rays. So the maximum number of points on the circle that can connect through reflection two given points A and B is four.

Al Haytham's solution

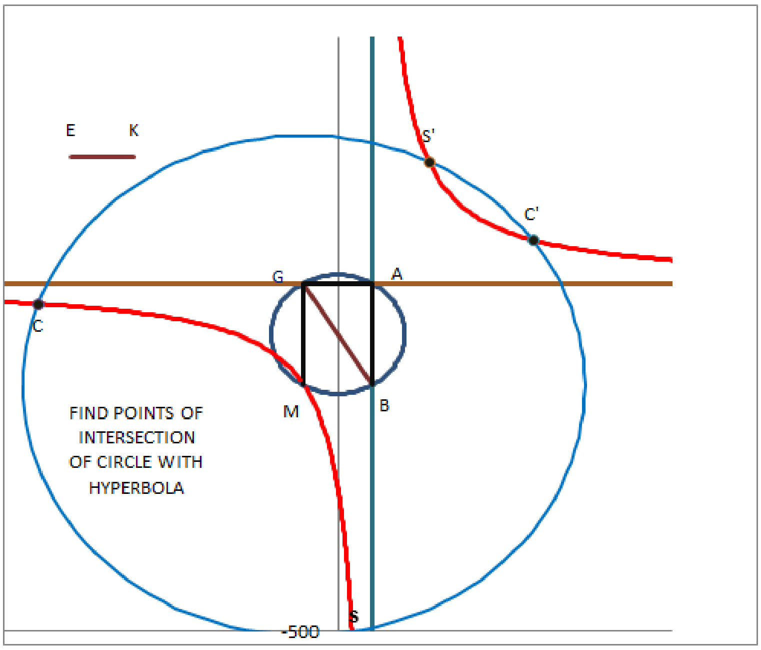
Al Haytham proved that the highest number of points that can join A and B are four. His proof is based on lemmas described in Sabra's work: *Ibn Al Haytham's lemmas for solving "AlHazens'" problem (Archive for the Exact Sciences Vol. 26 December 1982)*



Software was developed to have an interactive way of following the construction and the proof of the lemmas. This software has the form either of visual basic projects from which we have exe files, or we developed excel files without makros. In each lemma first we present the construction of the desired result and then the proof of the lemma. To find the 4 points we use the 4th lemma. This lemma in turn uses the 3rd lemma which in turn uses the 1st or second.

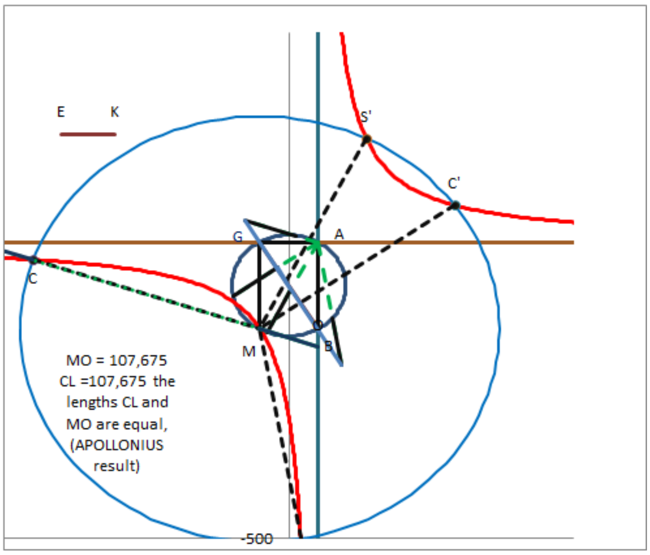
1st lemma:

Let circle *ABG* , with diameter *GB*, be known *;* let *GB* be produced the side of G; let line *KE* be given and let point A be given on the circumference of the circle. We wish to draw from A a line, as *AHD*, so that the part of it that lies between the diameter and the circle--such as *HD--is* equal to line *KE*.



The steps for the solution of this lemma are presented in the excel file "Al Haytham's first lemma"
In the first step we join the line BA, GA
In the 2nd and 3rd step we draw the GM parallel to BA and then we draw the hyperbolae which pass through M and have as asymptotes the lines GA, BA.
In the 4th step we draw the circle with center M and radius $MC=BG^2/KE$.
In the 5th step we find the points of intersection of the circle with the

hyperbolae. These points are designated as C, C’, S, S’.
In this excel file the Newton method of finding the first root was used for a fourth order equation:
$F(y) = y^4 - 2 * My \cdot y^3 + |M|^2 - Mc^2 \cdot y^2 - 2 * Mx^2 \cdot St \cdot y + St^2 2 = 0$, where $St = My \cdot Mx$
We divide the Fy by (y-y1) where y1 was the first root and we get a third order equation:
The fourth order constants are:
syntEL1(0) = 1: syntEL1(1) = -2 * Mx: syntEL1(2) = $|M|^2 2 - Mc^2$: syntEL1(3) = $-2 \cdot My \cdot Mx \cdot St$: syntEL1(4) = St^2
The third order equation is: $Tr(x) = Tr_0 \cdot x^3 + Tr_1 \cdot x^2 + Tr_2 \cdot x + Tr_3 = 0$, The Tr_i are found by:
$Tr_0 = 1$, $Tr_i = xx1 * Tr_{i-1} + syntEL1(i)$ where xx1=St/y1
We divide Tr(x) by x-xx2 where xx2 is the root of Tr(x) and we get a quadratic equation:
$Synt_0 \cdot x^2 + Synt_1 \cdot x + Synt_2 = 0$, where $Synt_0 = 1$
$sYNT_1 = xx2 * sYNT_0 + Tr_1$, $sYNT_2 = xx2 * sYN_1 + Tr_2$.
This quadratic equation determines if the Al Hazen’s problem has 4, 3 or 2 solutions. We calculate the discriminant diakr = $sYNT_1^2 - 4 \cdot sYNT_0 \cdot sYNT_2$. So if diakr>0 we get two more roots, if diakr=0 we get one more root and if diakr<0 there are no more real roots.
The same procedure is used in all Lemmas.
The next steps are used to determine the lines that have the length H.

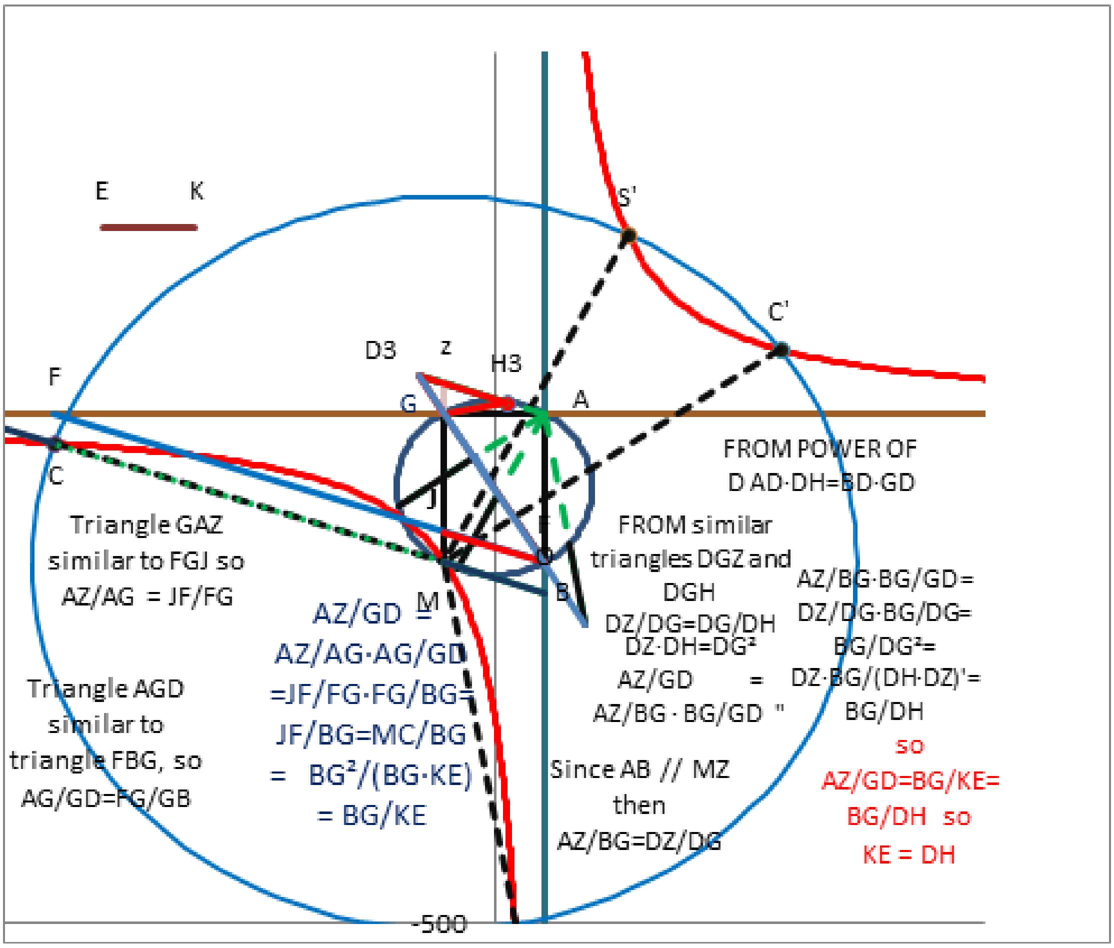


DETERMINATION OF THE LINES WITH LENGTH H

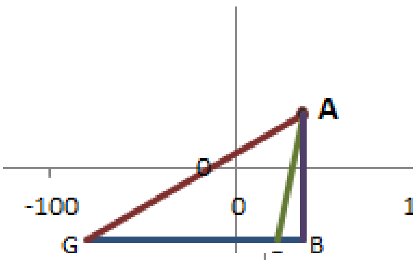
Step 6: Draw lines from point M to the points of intersection
Steps 7,8,9 Draw from point A lines that are parallel to the lines drawn from A that intersect the diameter BG and the circle. The lines HiDi have all the same length =EK.

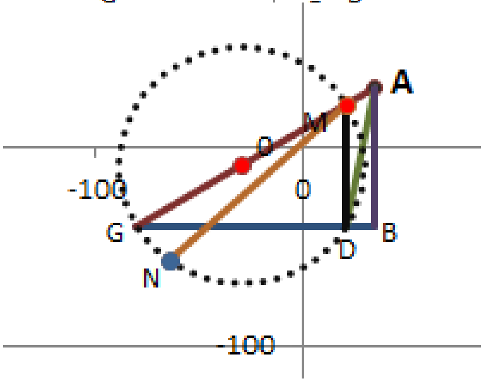
PROOF OF THE LEMMA

Step 10 We use Apollonius result (see Sabra 1982) The lengths LC and MO that are cut from the extension of the line MC to the point of intersection of the asymptote AG and the extension of MS to the asymptote AB are equal.
Step 11 We draw BF//ML, then ML= BF since MLBF parallelogram, since MG//AB MO=BJ
Step 12. We find the point J where BF meets BM
Step 13 we find the point where the line AH1 cuts the line MG, Angle ZHG = Angle GBA = Angle BGM = Angle ZGD. The first two are equal because BGHA is inscribed in the circle.
Step 14: Triangle GAZ similar to FGJ so AZ/AG = JF/FG,
Step 15: Triangle AGD similar to triangle FBG, so AG/GD=FG/GB
$AZ/GD = AZ/AG \cdot AG/GD = JF/FG \cdot FG/BG = JF/BG = MC/BG = BG^2/(BG \cdot KE) = BG/KE$ (1)
Since AB // MZ then AZ/BG=DZ/DG
FROM POWER OF D $AD \cdot DH = BD \cdot GD$
STEP 16: FROM similar triangles DGZ and DGH DZ/DG=DG/DH so $DZ \cdot DH = DG^2$
$AZ/GD = AZ/BG \cdot BG/GD$
$AZ/BG \cdot BG/GD = DZ/DG \cdot BG/DG = BG/DG^2 = DZ \cdot BG/(DH \cdot DZ) = BG/DH$
so AZ/GD=BG/KE= BG/DH so KE = DH



LEMMA 3





In the triangle ABG, B is a right angle, and D a point on BG,or on its extension toward B. It is required to draw from D a line that cuts the hypotenuse in a point, as T , and AB or its extension in another point, as K such that TK/TG= a given ratio E/Z

.

1st step

2nd step We draw AD (according to the value of DX = LEMMA3.delatTHESI.Value * (Bx - GX) / 100 + GX where delatTHESI is a scrollbar)

3rd step We draw DM//AB

4th step We find the center of GM.

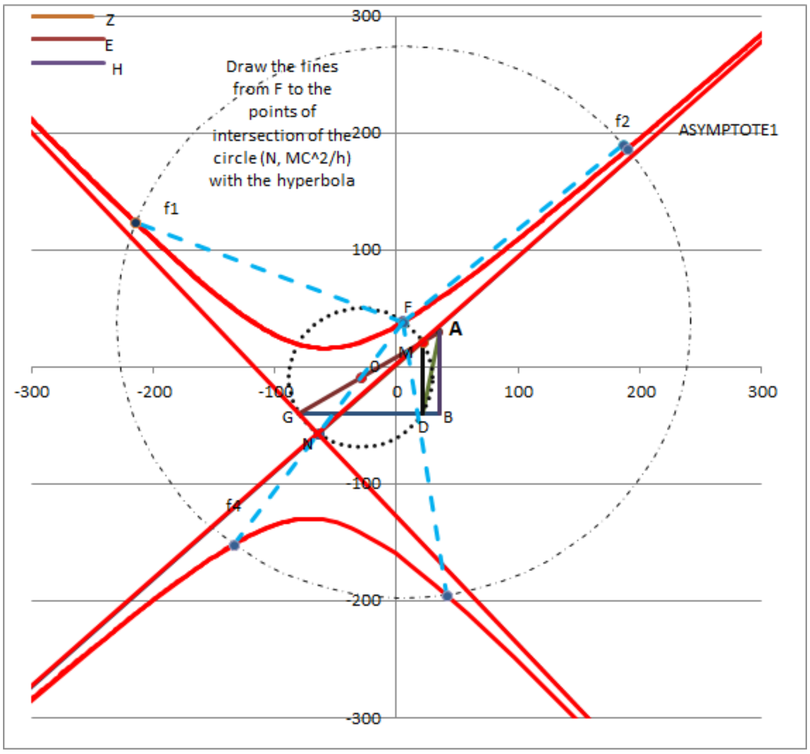
5th step We draw the circle with center the middle of GA and radius MG/2

6th step: We draw the angle NMD=angle DAG

7th step: Construct line H such that AD/H = E/Z (the given ratio)

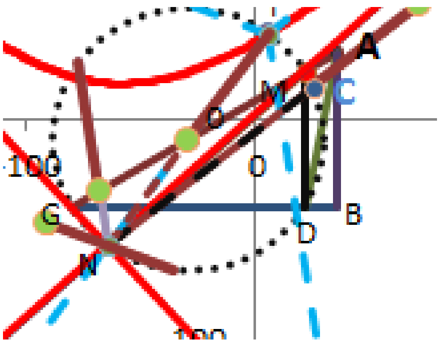
8th step: DRAW POINT F: Fx=Mx-(Nx-Gx), FY=MY+(GY-NY)

9th step: Construct the circle with center F and radius MN²/h



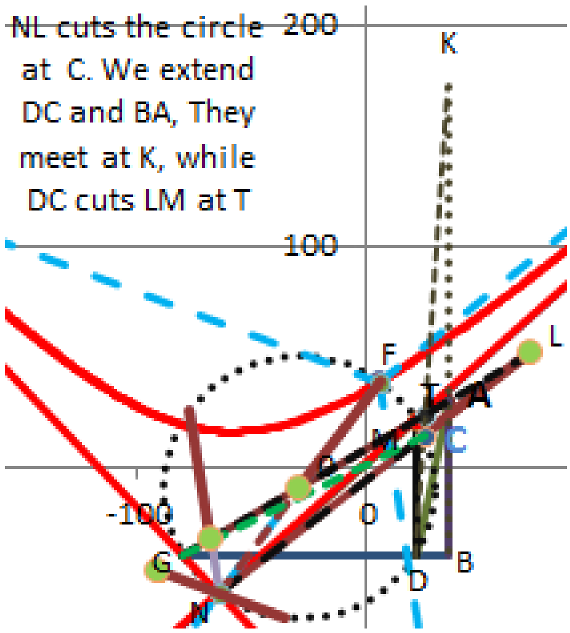
10th step: Construct ASYMPTOTES through N (LINE NM AND THE PERPENCIDULAR)
11th step: CONSTRUCT THE HYPERBOLA PASSING THROUGH F WITH THE GIVEN ASYMPTOTES
12th step: Find the intersection of the hyperbola with the circle (N, MC^2/h)
13th step: Draw the lines from F to the points of intersection of the circle (N, MC^2/h) with the hyperbola.
14th step and 15th step: Draw the lines from N that are parallel to the lines from F. According to the first lemma the lengths of the lines between the diameter GM and the circle with diameter GM are equal = H. Where h = AD * Z / E Z = 50: E = 60 (see diagram for Step 12)
In the diagram these lengths are presented by the thick lines.
Step 16: Let NL be one of these lines. C is the point where this line cuts the circle.

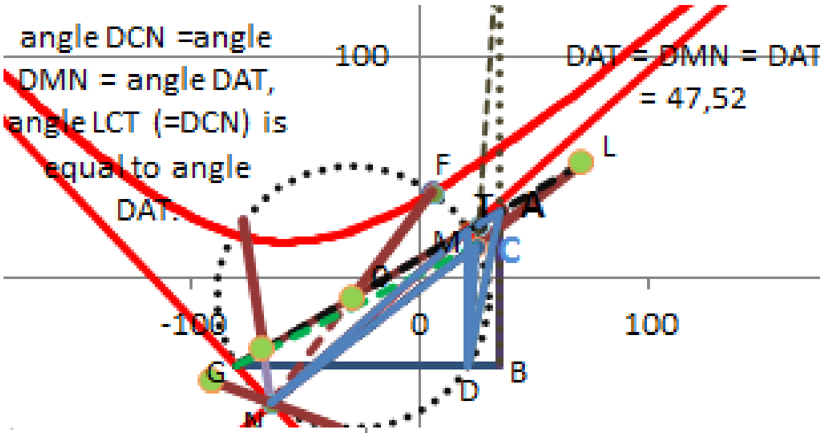
Step 12



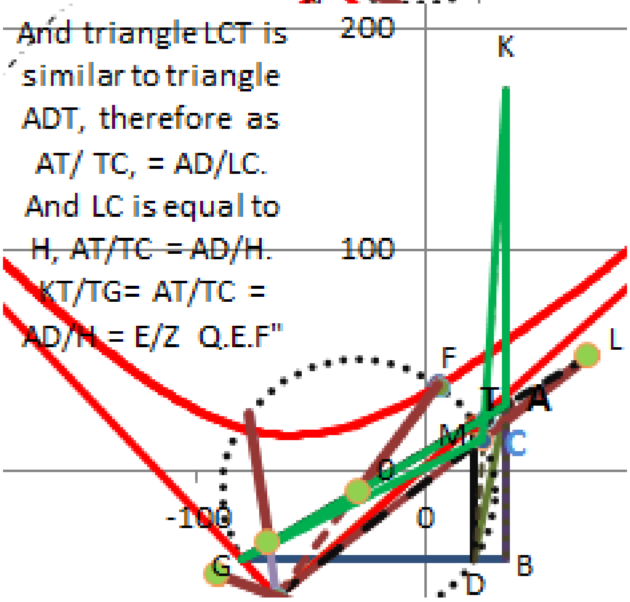
We extend DC and BA:

NL cuts the circle at C. We extend DC and BA, They meet at K, while DC cuts LM at T
Join GC.
Step 17: Angle GCD = Angle GMD = Angle GAB. The first two angles are equal because they are inscribed in the same arc. Angle GAB =angle GMD because GM//AB.
Step 18: Therefore angle GCT (= π-GCD) is equal to angle TAK (= π- GAB); but angle CTG is equal to angle ATK; therefore if line CT is produced in a straight line, it will meet line AK at an angle equal to angle TGC. Angle AKT= Angle TDM = Angle TGC.
Step 19: triangles AKT, CGT similar, angle TKA=angle TDM because DM//KA, angle CGM is equal to CDM as they are on the same arc.





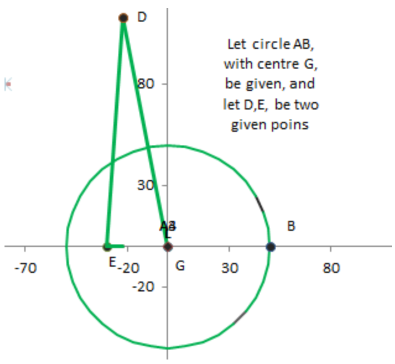
Step 20: angle DCN =angle DMN = angle DAT (=DAG see step 6), angle LCT (=DCN T is on DC) is equal to angle DAT.
Step 21: And triangle LCT is similar to triangle ADT, therefore as AT/ TC, = AD/LC. And LC is equal to H, AT/TC = AD/H. KT/TG= AT/TC = AD/H = E/Z Q.E.F



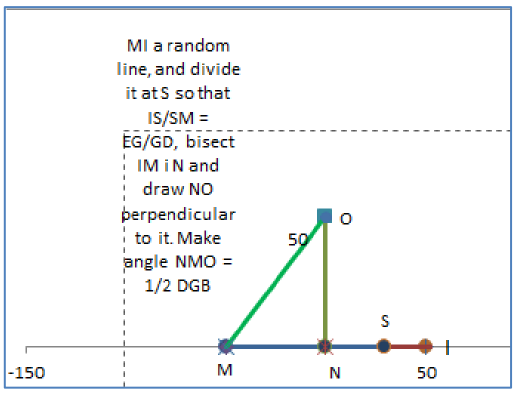
LEMMA 4

The most important is lemma 4 which states: *let circle AB, with center G, be given, and let D, E be two given points; we wish to draw from E, D, two lines like EA, DA, so that a line drawn tangentially to the circle, such as AH, will bisect angle EAD.*

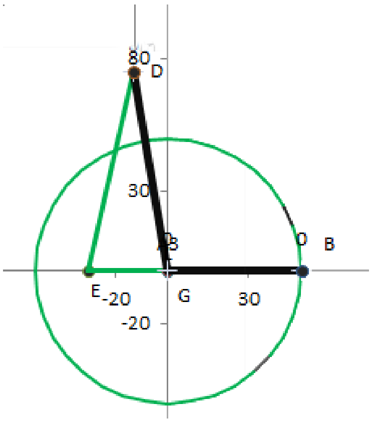
His proof is quite involved and to follow it. One must study first the lemmas 1 and 3. The proofs are given in Sabra (1981) and are presented by software in www,kyriakosxolio.gr

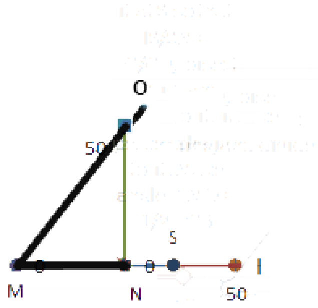


Step1: Let circle AB, with center G, be given, and let D,E, be two given points.
Step2: Join GD, GE, ED; and produce EG in a straigth line to B.
Step 3: MI a random line, and divide it at S so that IS/SM = EG/GD, bisect IM in N and draw NO perpendicular to it. Make angle NMO = 1/2 DGB



Auxiliary graph



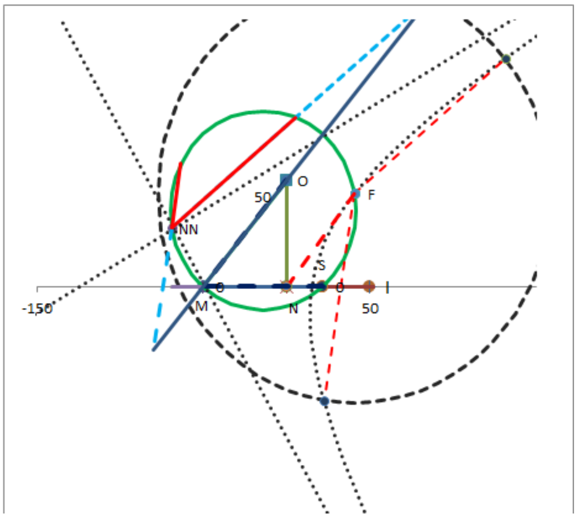


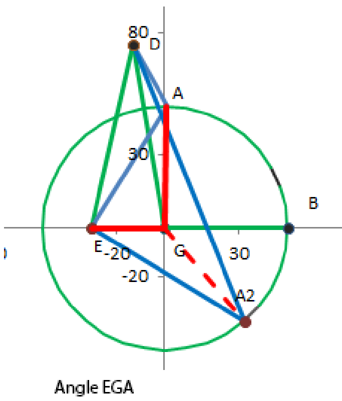
Angle NMO = 1/2 DGB

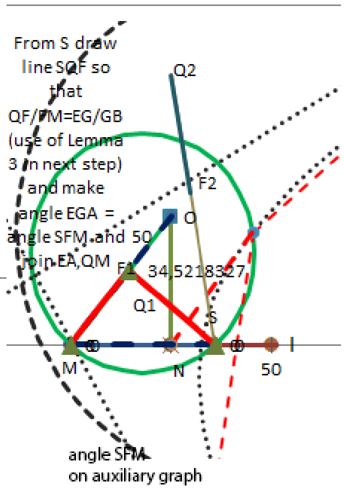
Step 4: From S draw line SQF so that QF/FM=EG/GB (use of Lemma 3 in next step) and make angle EGA = angle SFM and join EA,QM.

We use the Lemma 3.

From S draw line SQF so that QF/Fm = EG/GB (Use of Lemma 3 ON AUXILIARY FIGURE). Draw angle EGA=SFM and join EA







Use of Lemma 3

Step 5, Step 6:

Make angle EAZ = angle QMS, produce AZ on the side of Z and make the ratio of AZ to ZK equal to the ratio of MS to SI which is the same as the ratio of DG to GE.

Step 7

Join EK and produce EL perpendicular to AK

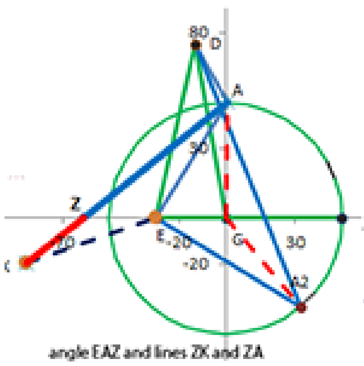
Step 8

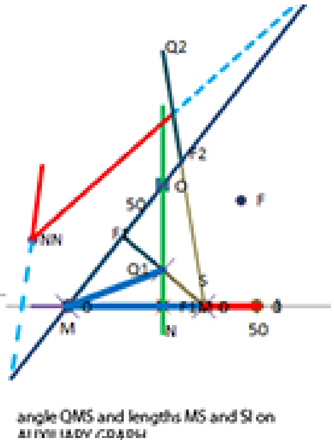
Draw AT parallel to EK.

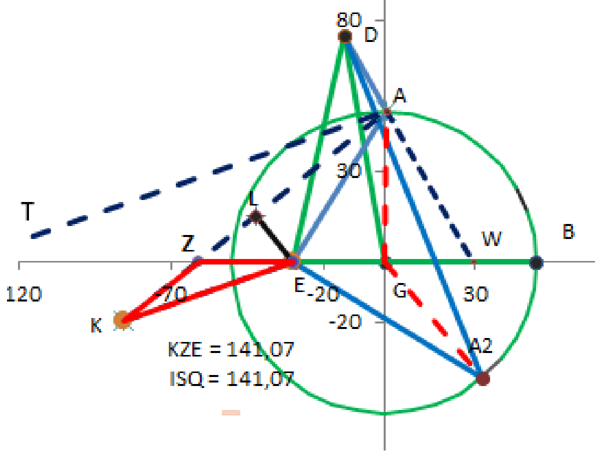
Proof of isosceles KAE: the angles at points A, E, K, Z, L will be equal to the angles at points M, Q, I, S, N, and, therefore, the triangles will be similar. From the similarity of triangles since QN = perpendicular in the middle of MS so EL is perpendicular to the middle of ZK and so KEA isosceles

Steps 9,10

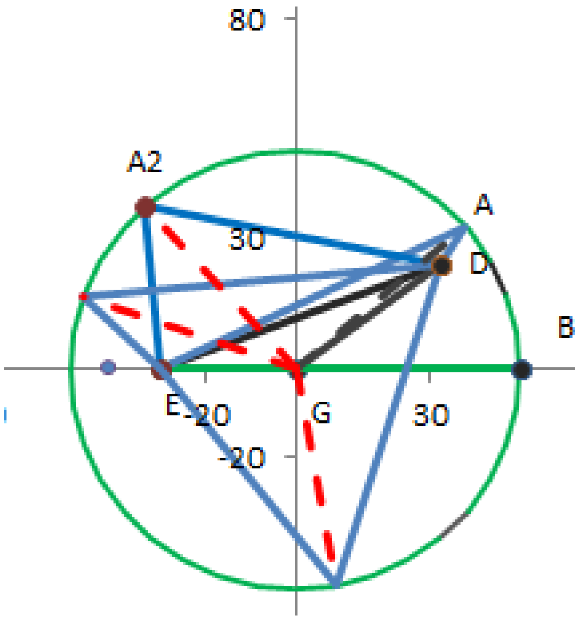
"Make GAW = GAE, therefore WAT=2 X GAZ (GAZ = NMO = 1/2 DGB) So WAT=DGW



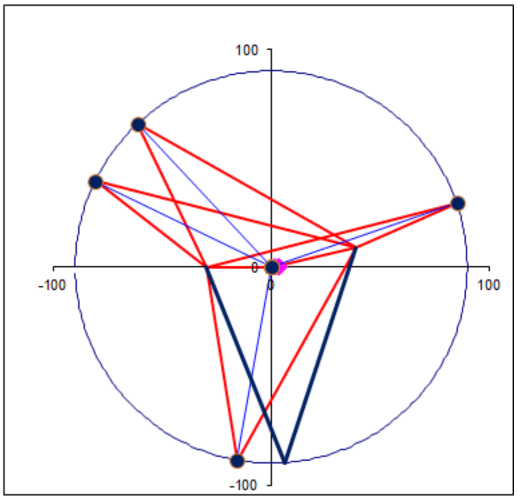


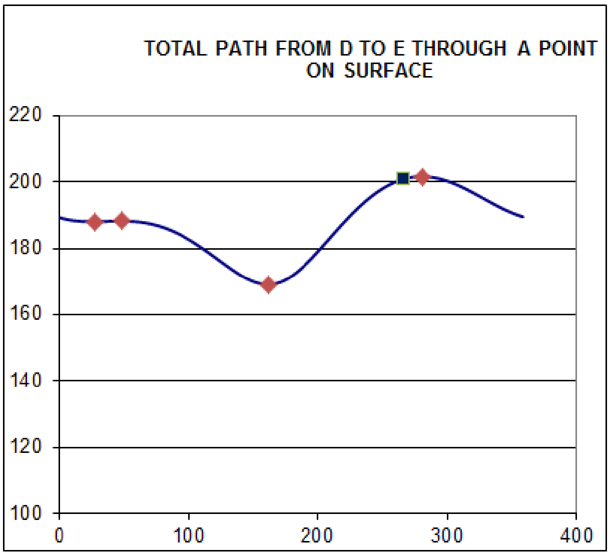


Step 11: We produce WA to cut GD at X. EG/GD=EA/AT=EA/AW x AW/AT = EG/GW x WG/GX=EG/GW x WG/GD or EG/GD=EG/GX. so GD is equal to the line cut off by WA from the line GD, thus WA will go through the point D.
Step 12 GE/GW=EA/AW (GE bisector of EAW)
Step 13 AT//EK so EA/AT=EZ/ZT and EZ/ZT=KZ/ZA which is = EG/GD (by construction) so EA/AT=EG/GD
TAD=π-TAW=π-DGW=EGD so angle TAD =angle EGD
Step 14 make GAH right angle
ZAG =1/2WAT = 1/2DGW
so ZAH= π/2-ZAG = π/2- 1/2WAT =1/2 TAD
Step 15 Thus angle ZAH is half of angle TAD, and angle ZAE is half of angle TAE, therefore angle EAH is half of angle FAD.
So GA which is the perpendicular to AH, bisects the angle EAW which is π-EAD
Lemma 4 can be used for finding the points where reflection can occur from A to D
We use for this case the Excel file "Use of Lemma 4"
Another Excel file that can be used is "AlHaytham no makros"
In this file a numerical solution is given with accuracy 1 degree. The graphs are compatible with the solution by Al Haytham presented here.
This also helps in teaching about Heron-Fermat's principle



Graph from "Use of Lemma 4"





Points of reflection on a sphere from Al Haytham no makros